Grinnell Arithmetic: A Conjecture

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Abstract

This paper introduces a conjecture, that there exists an alternative system of arithmetic which is both internally consistent and physically meaningful, such that a number set analogous to the set of real numbers is closed under all arithmetic operations, particularly including taking the root of a number to an even-numbered degree. That is, within this alternative system of arithmetic there would be no need to appeal to imaginary or complex number sets. Furthermore, this conjecture includes that such a system of arithmetic is shaped by the following two conjectured postulates: 1) All numbers are absolute values, fundamentally without sign, which exist in a superposition of sign spaces. Sign, and by extension the difference between addition and subtraction, is metadata attached to a number, expression, or function and is not an intrinsic part of the number, expression, or function itself. 2) All numbers are n-areas, that is, areas or volumes of an arbitrary number of dimensions, n, including n=1, whereby n is metadata attached to the number. Multiplication is not repeated addition, but instead is the calculation of a product n-area with a higher n. This proposed alternative system of arithmetic, with all of the attributes described above, shall be referred to as **Grinnell Arithmetic**.

Background: Arithmetic and Number Sets

"Arithmetic" is a top-level division of mathematics which deals with the basic operations which can be performed on numbers, primarily addition, subtraction, multiplication, and division, but also including expansions of these such as exponentiation, taking of roots, logarithms, and percentages.

Some of the basic terminology related to the arithmetic operations is used in this paper. The output of multiplication is a "product," while the inputs are "factors." The output of a division is a "quotient," such that the "dividend" is divided by the "divisor." Alternately, a quotient can be expressed as a "fraction," in which case the dividend is referred to as the "numerator," and the divisor is referred to as the "denominator." In this way, division is seen as the multiplication of the dividend by the reciprocal of the divisor, where the "reciprocal" of a divisor is the fraction made by dividing the number one by the divisor.

Further terminology includes the "exponent," which is the power to which a number is raised. For example, in the expression x^2, the exponent is two. The "degree" when taking the root of a number is the power to which the root would be raised to obtain the original number. For example, the degree of a square root is two since raising the square root to the power of two will result in the original number for which the square root was taken. The term "degree" can also refer to a polynomial function, where it is equal to the highest exponent found in the polynomial.

A plethora of "number sets" are defined where the numbers which are members of these sets have some particular property or properties in common. The principle number sets used in mathematics are: integers, rational numbers, irrational numbers, real numbers, and complex numbers.

"Integers" are the negative and positive numbers that are not fractions, including zero, for example: -2, -1,0,1,2. The set of "rational numbers" are those numbers which can be expressed as fractions or as decimals which terminate or repeat, for example: ½, -3.25, 3.66…, and contains all members of the set of integers. "Irrational numbers" are those which cannot be expressed with precision as a fraction and have decimal representations which never terminate and do not repeat, for example: 2.718…, $\sqrt{2}$, π. The set of "real numbers" contains all members of the sets of integers, rational numbers, and irrational numbers. Complex numbers are described later.

Background: Sign Rules for Products in Conventional Arithmetic

Conventional arithmetic allows a number to be subtracted from a smaller number, resulting in a negative number. These negative numbers can then be multiplied by other numbers. A positive number multiplied by another positive number results in a positive product. A negative number multiplied by another negative number also results in a positive product. A negative number multiplied by a positive number, in either order, is the only way to result in a negative product.

Background: Imaginary and Complex Numbers

Given the sign rules for products stated above, a negative number multiplied by itself yields a positive square, while a positive number multiplied by itself also yields a positive square. Under these rules, there is no number in the set of real numbers that can be squared to yield a negative square.

None the less, it sometimes occurs in conventional algebra that we are required to calculate the square root of a negative number. Since no real number can be squared to yield a negative number, the square root of a negative number cannot be expressed as a real number. This is what is meant to say that the set of real numbers is not "closed" with respect to the operation of taking a square root. This is true of taking the root to any even-numbered degree, for example, taking the fourth root, sixth root, etc.

The conventional way around this is to define an "imaginary unit" such that it is the theoretical number that can be squared to yield negative one. The imaginary unit, therefore, equals the square root of negative one, and is given the symbol *i*. The imaginary unit is the basis for the set of "imaginary numbers," which are comprised of a real number multiplied by the imaginary unit. The more general set of "complex numbers" are comprised of a real number added to an imaginary number. The set of complex numbers, which includes all real numbers and imaginary numbers as subsets, is closed with respect to taking the root of any degree.

Background: Internal Consistency, Physical Meaningfulness

All of mathematics can be said to be a self-referencing tautology; that is, it is based on a set of axioms which are considered to be self-evident, and theorems are drawn deductively from these axioms without any proof coming from outside of the system of mathematics itself. None the less, sound mathematics is "internally consistent," meaning that it does not contradict itself, and if one applies the theorems of mathematics reliably, any solution obtained will reliably be in agreement with all other solutions derived from the same axioms and theorems.

However, it is not entirely accurate to dismiss mathematics as a tautology, since solutions found by the application of mathematics tend to have "physical meaningfulness;" that is, such solutions tend to agree with observations about physical reality. For example, if one were to place two apples into an empty bag, and then later place three more apples into the bag, basic arithmetic would predict that there will be five apples in the bag. Counting the apples will verify this prediction. More importantly, however, accepted theories in the physical sciences are universally expressed in mathematical formulas, and the predictions made by such formulas are universally verified by experiment, at least to the level of precision allowed by the experiment.

It is important not to read too much into the name given to the imaginary unit, or to the imaginary component of complex numbers, since they are essential to the theories of many scientific and technical fields. Perhaps the most striking example is quantum mechanics – the theoretical basis for numerous real technologies including electronics, lasers, and nuclear reactors – where all of the wave functions which describe sub-atomic particles are complex functions. Other fields which rely on complex numbers include: signal processing, control theory, fluid dynamics, electrical engineering, and cartography. As such, complex numbers exhibit an extraordinary level of physical meaningfulness, or put another way, they get results in the real world.

The fact that conventional mathematics is both internally consistent and physically meaningful does not signify that it is the only possible system of mathematics which can express these traits. It only means that if one applies the conventional set of axioms and theorems reliably, one will get consistent results which sometimes correspond to some observable features of reality. It in no way obviates the possible existence of other viable sets of axioms and theorems.

Background: Multiplication as Repeated Addition

In conventional arithmetic, multiplication is generally treated as repeated addition. For example, five times three is equivalent to five added into a fresh running total three times. The sign rules for products in conventional arithmetic make sense in this context. For example, multiplying five by negative three is the equivalent of doing the reverse of adding five into a fresh running total three times, that is, subtracting five three times, yielding a negative product. Likewise, multiplying negative five by three is the equivalent of adding negative five into a fresh running total three times, also yielding a negative product. Meanwhile, multiplying negative five by negative three is the equivalent of doing the reverse of adding negative five into a fresh running total three times, that is, subtracting negative five three times, yielding a positive product.

Note that in this repeated addition model, all the factors and products exist on the same singular number line, or dimension. Of course, multiplication in conventional arithmetic can also be treated as the calculation of a two-dimensional area, but this only applies to the special case where an area is the sought after solution, not as the fundamental nature of multiplication. Also, to consider a negative length or a negative width is nonsensical, so there can be no consideration of sign when applying multiplication to calculate an area.

Background: Dimensional Analysis

In engineering and science, "dimensional analysis" is a conventional technique used as a way to check the plausibility of derived formulas, as well as a method of converting between units of different

measuring systems, such as kilometers and miles. It defines the concept of a "fundamental dimension," which is a type of physical quantity which is basic and therefore cannot be expressed as a combination of other types of physical quantities; examples are: length, mass, time, monetary units, and electric charge. Dimensional analysis makes a distinction between "base units," which are measuring units related to a fundamental dimension, and "derived units," which are measuring units that can be expressed as products or ratios of base units, for example: kilometers per hour and amperes, which is the equivalent of coulombs of electric charge per second.

Dimensional Analysis is also used to determine when it is meaningful for two quantities to be compared, equated, added or subtracted; for example, it is nonsensical to ask whether one dollar is more, the same, or less than one kilometer, or to add one dollar to one kilometer. Dimensional analysis is used to check the plausibility of derived formulas by ensuring that the composition of fundamental dimensions is the same on each side of the equation. For example, a formula intended to describe the energy of a system – where energy is defined as the potential to accelerate a mass across a distance – must have variables as inputs which comprise the correct composition of variables for length, mass, and time.

Background: Functions, Domain, and Range

A "function" is an equation which assigns a value to a "dependent variable" – the output of the function – as the evaluation of an expression containing one or more "independent variables" – the inputs to the function. Strictly speaking, to be considered a function an equation must result in one and only one value for the dependent variable given a particular combination of values for the independent variables; however, it is quite common in mathematics to work with equations that do not meet this requirement.

The "domain" of a function is the set of numbers which are relevant to be used as values for the independent variables, both in terms of the type of numbers, such as integers or complex numbers, and their location on the number line. Relevancy here is determined by the nature of the function and the way it is defined. The "range" of the function is the same concept when applied to the dependent variables. The domain of a function can be broken down into specific subsets called "intervals;" for example, the subset of real numbers between one and five can be specified as an interval of the domain of a function.

Background: Asymmetries of Conventional Arithmetic

There are at least two notable asymmetries resulting from the sign rules of conventional arithmetic. The fact that these asymmetries exist in no way impairs the fact of conventional arithmetic being internally consistent, or even physically meaningful. They are curious, however, and detract somewhat from a sense of "elegance" in the mathematics. That is, they treat one thing as being different from another thing, when there is no deeply fundamental difference between the two things.

The first asymmetry is that there is an absolute value function that converts any negative number to a positive number, but leaves positive numbers unchanged. This implies that there is somehow a fundamental difference between positive and negative real numbers, when the only difference, at least at the level of pure mathematics, is that they are on opposite sides of zero on a common number line. The second is that taking the root of a real number to an even-numbered degree

must result in a complex number, while taking the root of a real number to an odd-numbered degree results in another real number. This implies that there is somehow a fundamental difference between even and odd numbers, when the only difference between even and odd numbers is the presence or absence of two as one of their prime factors. There are other asymmetries, such as those related to the exponents in polynomial functions and those related to logarithms; but they all tend to make these same questionable distinctions between positive and negative numbers, or between even and odd numbers in the context of exponentiation.

Grinnell Arithmetic: Introduction, Metadata

Grinnell Arithmetic is conjectured to be an internally consistent and physically meaningful novel system of arithmetic which eliminates the need for imaginary and complex numbers. As stated above, imaginary and complex numbers exhibit an extraordinary level of physical meaningfulness; so their elimination is not sought after because they fail in any practical sense, but because there may be something better. The central premise is that the imaginary unit works as well as it does because *it is a useful placeholder for missing information*, and Grinnell Arithmetic seeks to prevent such information from being lost in the first place, or at least to provide alternative methods for dealing with such missing information. In this sense, it is accurate to consider Grinnell Arithmetic to be primarily an information management protocol.

Essentially, this is to be accomplished by: 1) attaching metadata to numbers, expressions, and functions, 2) allowing for numerous permutations of expressions and functions, and 3) developing rules to manage such metadata and permutations, and to map these onto real-world situations. As will be seen, the amount of such metadata will become very large very quickly, as will the number of permutations. As such, it is expected that the application of Grinnell Arithmetic will be virtually impossible without the use of computer software. There are some problem sets in physics and mathematics that are very difficult to work with directly using algebra or calculus, for example, those that involve the Navier-Stokes equations used in fluid dynamics and those that involve the calculation of a definite integral for functions with two or more independent variables. It is common practice in these situations to use "computational methods" – number crunching – performed by computer software to calculate numerical solutions. *This is not what is meant here.* The point being made is that even to work with equations *algebraically* under Grinnell Arithmetic will almost always require the use of computer software to manage all of the metadata and permutations.

Grinnell Arithmetic: The Grinnell Number Set

Grinnell Arithmetic is more than just a conjectured novel method for performing arithmetic operations; it also defines a novel number set. This set of **Grinnell numbers** is analogous to the real numbers in that it includes the equivalents of the integers, rational numbers, and irrational numbers. It differs from the real numbers in that the numbers are neither inherently positive nor negative. It also differs from the real numbers in that Grinnell numbers do not all share a common number line, or dimension. Both sign and dimensionality are metadata attached to the numbers, and not intrinsic to the numbers themselves.

One may also consider the subsets of Grinnell numbers which are similarly analogous to any of the other conventional number sets, with the added constraints that they are inherently devoid of sign and may be described by multiple dimensions. These could be called **Grinnell integers**, **Grinnell**

rationals, and **Grinnell irrationals**. In the following discussion, the distinction is not made between these conventional sets and the Grinnell sets, since such a distinction can be inferred from the context.

Grinnell Arithmetic: First Conjectured Postulate; Superposition, Permutations, and Sign Space

The first conjectured postulate of Grinnell Arithmetic is that all Grinnell numbers are absolute values or magnitudes, fundamentally without sign, which exist in a superposition of sign spaces. In conventional arithmetic, the subtraction of a number is equivalent to the addition of the opposite of that number; that is, $a - b = a + (-b)$. Consequently, for Grinnell numbers to be inherently devoid of sign, the concepts of addition and subtraction must be adjusted.

Each simple one-dimensional Grinnell number is considered to exist in two sign spaces simultaneously; that is, the negative version and the positive version exist separately but overlapping. This simultaneous existence of different versions of a number, as well as different versions of an expression or a function, will be referred to as **superposition**. In any expression in which there would conventionally be an addition or subtraction between two numbers, there is instead a summation resulting in four **permutations** of the expression; that is, given two simple one-dimensional Grinnell numbers, a and b, summation results in the four possible combinations of positive a, negative a, positive b and negative b – all existing in superposition. As discussed later, permutations are also generated during multiplication operations, and when expressions are equated to functions.

For simple one-dimensional Grinnell numbers, the term **sign space** will refer to a specific choice of the positive or negative version of the number; whereas for expressions and numbers derived from expressions, sign space refers to a specific permutation. For functions, sign space refers to the specific permutations of the ranges of the function which are either positive or negative, over specific intervals of the domain. Each addition of a dimension to a number, discussed later, will double the number of sign spaces. Permutations and sign spaces are metadata attached to a number, expression, or function.

The concept of such a superposition of possible sign states is not entirely absent from conventional mathematics. It is found, for example, in the quadratic formula and is given the symbol: $\pm$, spoken as "plus or minus." When solving the quadratic formula, it is necessary to solve it twice, once for the case where plus or minus is considered positive, and again for the case where it is considered negative. Generally, two different solutions are found.

Also note that when solving the quadratic formula for real-world problems, it is common for one of the solutions to be nonsensical in the context of the problem, although being mathematically correct. These nonsensical solutions are called "extraneous roots," and by definition are not physically meaningful. Similarly, many, if not most, of the permutations of an expression within the Grinnell Arithmetic framework will not be physically meaningful; these will be called **extraneous permutations**. The goal of Grinnell Arithmetic is to retain and manage information, not to systematically prejudge whether or not a given piece of information is meaningful.

Grinnell Arithmetic: Perspective, Function Level Rules

To further illustrate the concept of superposition, consider the example of a transaction between a buyer and a seller involving the sale of an item. Both the buyer and the seller begin with an initial bank account, and end with a final bank account after the transaction; the selling price of the item remains constant. In Grinnell Arithmetic, the expression to describe the final bank account for both the buyer and the seller would be: initial bank account sum selling price. Note that the same expression applies to both the buyer and the seller. In the case of the buyer, the permutation which is physically meaningful is the one where initial bank account is positive and selling price is negative. In the case of the seller, the physically meaningful permutation is the one where both initial bank account and selling price are positive. The other two of the four possible permutations are extraneous for both buyer and seller, but for the two permutations just described the question of whether or not the permutation is extraneous depends on perspective. For the buyer, one of those two permutations is extraneous, while for the seller, the other permutation is extraneous.

This concept is extremely important, and will be called **perspective**, defined as relating to the sets of permutations which are physically meaningful in a particular situation, such as for a particular observer. For many of the formulas which apply to real-world problems, including theories in physics, the choice of addition or subtraction appears to be prima facie obvious and exclusive. However, it is hoped that by the application of the open-ended framework of Grinnell Arithmetic, hidden connections between seemingly unrelated problems will become apparent. Again, the goal of Grinnell Arithmetic is to retain and manage information, not to systematically prejudge whether or not a given piece of information is meaningful.

Note that if the above example is expressed as a function, that is, final bank account *equals* initial bank account sum selling price, the function itself also has permutations that exist in different sign spaces. These permutations are related to specific intervals in the domain of the function which result in different signs for the ranges that are expressed in these intervals. In particular, in the domain interval where the buyer's initial bank account is less than the selling price – indicating that the buyer incurs debt during the transaction – the physically meaningful permutation for the final bank account function from the buyer's perspective will be in a range with a negative sign space. In the domain interval where the buyer's initial bank account is greater than the selling price, the physically meaningful permutation for the final bank account function from the buyer's perspective will be in a range with a positive sign space.

In general, the term **function level rules** will refer to metadata attached to a function which provide information about which sign spaces for the range of a function are either physically meaningful or extraneous in specific situations, over specific intervals in the domain. It is at the function level, where an expression is equated to its evaluation, that the question of perspective is managed.

As another example, consider the graph of the function $F(x) = x^2$, a second degree polynomial. Conventionally, this function describes a parabola with both tails pointing upward. The function $F(x) = x^3$, a third degree polynomial, conventionally describes a curve similar to a parabola, but with one tail pointing up and the other pointing down. In Grinnell Arithmetic, both functions would have permutations with each of the four possible combinations of left or right tails pointing up or down. In the case of the second degree polynomial, the conventional parabola with both tails pointing the same way is especially physically meaningful. A parabolic dish based on such a function can be used to focus sound or electromagnetic waves, a cable spanning two poles will take the shape of a parabola, and an object on a ballistic trajectory bounded within a gravitational field will trace out a parabola. Perhaps a second degree polynomial with one tail pointing up and the other pointing down would also

have interesting features in some contexts. Perhaps a third degree polynomial with both tails pointing in the same direction would have interesting features. In Grinnell Arithmetic all such permutations can be managed and considered, while function level rules narrow the field down to legitimate solutions.

Grinnell Arithmetic: Second Conjectured Postulate; n-Areas, Substance

The second conjectured postulate of Grinnell Arithmetic is that all Grinnell numbers are n-areas of an arbitrary number of dimensions, n, whereby n is metadata attached to the number or to an expression or function which evaluates to a number. It can be seen that area and volume are the same type of thing, differing only in the number of dimensions. To expand this concept to any number of dimensions, including one, Grinnell Arithmetic will use the generalized term **n-area**, where n refers to the number of dimensions associated with the area or volume. Multiplication of Grinnell numbers can be thought of geometrically as the calculation of an n-area, such that the number of dimensions of the product equals the sum of the numbers of dimensions of the factors. A corollary of this concept is that multiplication of Grinnell numbers is explicitly not repeated addition, since repeated addition occurs within a single dimension. Grinnell numbers have both magnitude and dimensionality, where the **magnitude** of the n-area represents the count of unit n-areas which fit exactly into the n-area, and the **unit n-area** is defined as an n-area with the same dimensions as a particular Grinnell number, and with a length of one unit on all edges.

As stated earlier, each simple one-dimensional Grinnell Number is considered to exist in two sign spaces simultaneously; that is, the negative version and the positive version exist in superposition. A product of two simple one-dimensional Grinnell numbers will therefore occupy four sign spaces in superposition. That is, given two one-dimensional Grinnell numbers, a and b, multiplication results in the four possible combinations of positive a, negative a, positive b and negative b. In this simple case, these sign spaces correspond to the four quadrants of the Cartesian plane. Generalizing further, the number of sign spaces for a product of Grinnell numbers is equal to twice the product's n.

Consider quadrant I of a conventional Cartesian plane with axes x and y, where the variables a and b are positive real numbers. Calculation of the area, a times b, can be interpreted as the calculation of the definite integral, over the interval between zero and a, of the function $y = b$. That is, the area can be considered to be a sum of infinitesimals of constant height. Given the commutative property of multiplication, and that multiplication of one-dimensional Grinnell numbers is viewed as the calculation of a 2-area, it follows that one-dimensional Grinnell numbers can be viewed as infinitesimals. These can be referred to as 1-areas, having a well-defined “length” in one dimension, but an infinitesimal “width” in any other dimension. The implication of this is that number lines in Grinnell Arithmetic are not mathematically one-dimensional. Instead, all number lines have an infinitesimal thickness in every possible dimension, of which there are an infinite number. Generalizing further, all n-areas of any n also have an infinitesimal thickness in every possible dimension.

Included in the metadata related to the number of dimensions, n, will be information related to the **substance** of each dimension, defined here. In a purely mathematical context, the substance of a dimension would simply be the variable which brought the dimension into existence. For example, for the expression $a \times b$, assuming that both a and b are 1-areas, the resulting 2-area would have one dimension with an a substance, and another dimension with a b substance. In a real-world context the substance of a dimension will generally be the fundamental dimension, as the term is used in

conventional dimensional analysis, which the dimension represents. For example, a 2-area describing a speed would have one dimension with a distance substance and one dimension with a time substance.

The proper approach to the division of Grinnell numbers is to view it as the multiplication of the dividend by the reciprocal of the divisor. The reciprocal of the divisor represents an n-area, the magnitude of which is generally less than the unit n-area. This approach results in an increase in dimensions, whereas a decrease in dimensions may seem more intuitively appropriate to the reduction of an area which a division represents. Two reasons follow as to why this approach is more sensible than attempting to perform division in a manner which would result in a reduction of dimensions.

First, consider a 2-area comprising the product of two 1-areas, for example, four and five. The magnitude of the 2-area is twenty, representing the count of the twenty unit squares which fit into the 2-area. This can be visualized geometrically as a rectangle made up of a grid of twenty squares arranged in four columns and five rows. If one were to slice up the 2-area into four slices in order to divide it by four, these slices would not be 1-areas; that is, they would not be infinitesimals. Instead, each of the four slices would still be a 2-area made up of a column of five unit squares, each column having a height of five units and a width of one full unit.

Second, in formulas used in real-world applications, it is common for division to describe a rate, in which case the numerator and the denominator represent different fundamental dimensions. For example, average speed is equal to a distance traveled divided by the time spent to travel the distance, say, kilometers per hour. Distance and time are fundamentally different types of physical quantities, so attempting to slice up a 1-area representing a distance into a number of pieces representing a time is meaningless (Lorentz transformations not withstanding). Furthermore, an attempt at reducing the number of dimensions for such a rate would result in a loss of information, which is undesirable within the Grinnell Arithmetic framework.

Loose Ends: Introduction

At the time of the writing of this paper, there are some fairly serious gaps in the basic development of Grinnell Arithmetic, where it is unclear to the author how these issues can be resolved. These will be discussed now, along with some wild speculations.

Loose Ends: Dimensional Agreement

For two Grinnell numbers to be compared, equated, added or subtracted, it seems like it would be necessary for there to be a one-to-one correspondence between the dimensions and their substances for both Grinnell numbers, just as in conventional dimensional analysis. This concept will be referred to as **dimensional agreement**. For real-world problems, where the substances directly correspond to fundamental dimensions, this may or may not be a problem depending on how complicated the metadata is for the two n-areas due to the retained information generated by the history of previous arithmetic operations. For purely mathematical formulations, where variables tend to be entirely generalized, it is unclear how dimensional agreement can be determined and managed.

A particular concern with respect to dimensional agreement involves the introduction of a Grinnell number into formulations where its history is not known or specified. That is, if it is not known which arithmetic operations may have produced the number, then it cannot be known which dimensions may or may not be attached to it.

Loose Ends: Exponentiation and Root Taking

Simple exponentiation in Grinnell Arithmetic could conceivably be treated as repeated multiplication. The term "simple" here is meant to indicate the case where the exponent is an integer. This would result in an increase in dimensions, all with the same substance. For example, for the expression a^3, three dimensions are generated such that all three have an a substance. For this simple case, there does not appear to be anything fundamentally wrong with this approach.

Likewise, to take the root of an n-area to an integer degree, a simple case can be considered where the Grinnell number to be rooted has the same number of dimensions as the degree of the root, and all of the dimensions have the same substance. In this simple case it appears that the rooting operation should result in a reduction of dimensions. For example, consider a 3-area Grinnell number that describes a right rectangular prism where the edge lengths are not the same – it is a box but not a cube – and all three dimensions have the same substance. The cube root of the 3-area will be the 1-area representing the length of a single edge of the cube which encloses the same volume as the original prism, but with the least possible surface area.

The problem is that it is unreasonable to restrict exponentiation of Grinnell numbers to integer exponents. In conventional mathematics it is a perfectly reasonable, and sometimes necessary, approach to treat the taking of the root of a number as equivalent to raising the number to an exponent which is the reciprocal of the degree of the root. This strongly implies that in Grinnell Arithmetic exponentiation and the taking of a root should be accomplished by exactly the same procedure, treated as exponentiation with rational number exponents. Also, it is unreasonable to expect that when it is desired to take the root of a Grinnell number, it will always have a number of dimensions equal to the degree of the root, all with the same substance.

Note that even though the specifics of taking a root is not clearly specified within the Grinnell Arithmetic framework at this time, it is still guaranteed that a number set analogous to the complex numbers will not be necessary, regardless of whether the degree of the root is even or odd numbered. This is because Grinnell Numbers are defined to be fundamentally without sign, and so the sign rules for products found in conventional arithmetic do not apply.

Additionally, it is unreasonable to restrict exponentiation of Grinnell numbers to rational number exponents. To raise a Grinnell number to an irrational number exponent, using the simple model of repeated multiplication described above, implies the creation of a dimension which is somehow not a full dimension, and the degree to which it is only a partial dimension cannot be expressed numerically with precision. It is unclear at this point how that could even be defined.

Still, it seems desirable that basic exponentiation and taking a root be unified as an exponentiation, allowing both rational and irrational exponents. This implies that taking a root will result in an increase in dimensions, just as with division.

There are a number of concepts of dimension in conventional mathematics which allow the number of dimensions to attain non-integer real values, such as the Hausdorff dimension and the Minkowski dimension. These are generally associated with fractal geometry, and are not necessarily dimensions in the conventional sense. However, a fractal geometric approach to exponentiation, using

natural logarithms, may be part of a solution to the problem of exponentiation of Grinnell numbers for both rational and irrational exponents.

Interestingly, a feature of the Minkowski dimension is a concept of "entropy numbers," which is analogous to the concepts of thermodynamic entropy and information-theoretic entropy. The author considers entropy to be the single most important concept in all of the physical sciences, and has long suspected that there exists a potential unification of thermodynamics and information theory under a more general theory of entropy.

Another speculation about a possible solution to exponentiation is to consider dimensions as having curvature. This would be along the lines of some form of non-Euclidean geometry and would represent a topological approach. Examples of this type of treatment in theoretical physics can be found in the curved spacetime of general relativity and the "curled up" dimensions of some of the various models known collectively as "string theory." Euler's number is an irrational number which has profound significance in many areas of mathematics and real-world problem solving; and it is the base of the natural logarithm which in turn can be defined as a definite integral of the hyperbolic function. Given this, and the fact that logarithms can be used to calculate roots of any degree, it seems likely that the specific curvature that would be most relevant to this approach would be hyperbolic.

A final speculation about a possible geometric solution to exponentiation is to consider that under Grinnell Arithmetic there may be no need to require that a new dimension added to an existing system of dimensions is normal to that existing system of dimensions. In conventional concepts of dimensional spaces, the axis of each dimension is normal – perpendicular – to all other axes in that system; for example, length, width, and height axes are all perpendicular to each other. Perhaps by including a variable in the metadata for an n-area, representing an angle at which the axis of a new dimension intersects with the origin of the existing set of dimensions, it may be possible for n-areas within the new set of dimensions to evaluate in a sensible way to geometric solutions for irrational exponents.

Loose Ends: Algebra, Calculus, and All Other Mathematics

The total body of knowledge comprising modern mathematics is overwhelmingly immense. The introduction of Grinnell Arithmetic presented here is not even complete as an arithmetic; the question remains as to whether it can function reliably as a basis for other divisions of mathematics as well. Among all of the necessary requirements for Grinnell Arithmetic to be internally consistent and physically meaningful, a critical one is that it accommodates all significant applications of algebra and calculus. In the writing of this paper, the author has not given much consideration to these issues. If anything is indeed to become of Grinnell Arithmetic, very advanced artificial intelligence must be developed in order to apply the techniques found in algebra, calculus, and all other mathematics simultaneously to all of the permutations of expressions and functions generated by Grinnell Arithmetic.

Loose Ends: Division by Zero

Since the process of division is different in Grinnell Arithmetic compared to conventional arithmetic, it is conceivable that under Grinnell Arithmetic division by zero will not continue to be undefined. If the reciprocal of a divisor represents an n-area with a magnitude less than the unit area,

the reciprocal of zero may represent a Grinnell number which has zero or maybe infinitesimal magnitude, yet retains some dimensionality. Division by zero may therefore simply equate to multiplication by zero, but with some special case treatment of the metadata related to the dimensions involved. Of course, consideration must be given to the assymtotic behavior which often results in functions at points of division by zero, and to the treatment of division by zero using limits.

Conclusion

When Isaac Newton set about to formulate his classical mechanics, the mathematics of the day was not sufficient; so he had to invent part of calculus. Likewise, when Albert Einstein set about to tie up the loose ends in his theory of special relativity by formulating general relativity, he required the use of a form of mathematics which was relatively new at the time – Riemannian geometry. It seems to be a common situation for advances in physics to be predicated on nearly simultaneous advances in mathematics.

Complex numbers are a powerful tool when used to describe the workings of quantum mechanics. Still, consider that in recent times theoretical physicists have expended intense effort in the search for a "unified field theory" which would unite general relativity and quantum mechanics as one theory. It is quite possible that the limiting factor in this search is an inadequacy in existing mathematics.

Relativity achieved its initial goal of explaining the Lorentz transformations by treating time as a dimension which is not separate from the dimensions of space, but united with them as part of a spacetime continuum. As mentioned earlier, quantum mechanics relies heavily on the use of complex numbers in its formulation. The standard method for working with complex numbers is by invoking the concept of the "complex dimension," a framework whereby the imaginary component of a complex number is treated as a magnitude in an imaginary dimension, and the real component is treated as a magnitude in one or more real dimensions. The imaginary and real components together then define a point in the resultant complex space. So, both relativity and quantum mechanics work, in part, by taking a view of dimensionality which transcends the basic notion of three isolated, Cartesian spatial dimensions. Although highly speculative, it is conceivable that a mathematical system which greatly expands on the role of dimensionality, and which takes more of a geometric approach – such as Grinnell Arithmetic – will more clearly show how the four dimensional spacetime of relativity and the complex dimensions of quantum mechanics relate to each other.

In summary, Grinnell Arithmetic is proposed to be at once an information management protocol, a number set, and a system of arithmetic operations. The central concepts are that all Grinnell numbers are absolute values, and all Grinnell numbers are n-areas. This treatment results in the generation of countless permutations of sign spaces, expressions, and functions existing in superposition, and necessitates the development of rules to manage these permutations. While there are serious gaps in the model at the time of writing, it is only fair to consider it to be at an embryonic stage, and it may well be true that the basic concepts described here represent the very humble beginning of a new tool for understanding our universe which is both internally consistent and physically meaningful. This conjecture is presented for the reader's consideration.